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Generalized Weierstrass representation for surfaces in terms of Dirac–Hestenes spinor field

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Abstract

A representation of generalized Weierstrass formulae for an immersion of generic surfaces into a 4-dimensional complex space in terms of spinors treated as minimal left ideals of Clifford algebras is proposed. The relation between integrable deformations of surfaces via mVN-hierarchy and integrable deformations of spinor fields on the surface is also discussed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The theory of integrable deformations and immersions of surfaces due its a close relationship with the theory of integrable systems at the present time is a rapid developing area of mathematical physics. One of the most powerful methods in this area is a Weierstrass representation for minimal surfaces [1], the generalization of which onto a case of generic surfaces was proposed by Konopelchenko in 1993 [2,3] served as a basis for the following investigations. So, the generalized Weierstrass formulae for conformal immersion of surfaces into 3-dimensional Euclidean space are used for the study of the basic quantities related to 2D gravity, such as Polyakov extrinsic action, Nambu-Goto action, geometric action and Euler characteristic [4]. This method is also intensively used for the study of constant mean curvature surfaces, Willmore surfaces, surfaces of revolution and in many other problems related with differential geometry [5–14]. A further generalization of Weierstrass representation onto a case of multidimensional Riemann spaces, in particular onto a case of 4-dimensional space with signature $(+, +, +, -)$ (Minkowski space–time) has been proposed in the recent paper [15].

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In the present paper we consider a relation between a Weierstrass representation in a 4-dimensional complex space $\mathbf{C}^4$ and a Dirac–Hestenes spinor field which is defined in Minkowski space–time $\mathbf{R}^{1,3}$. Dirac–Hestenes spinors were originally introduced in [16,17] for the formulation of a Dirac theory of electron with the usage of the space–time algebra $C\ell_{1,3}$ [18] in $\mathbf{R}^{1,3}$ (see also [19]). On the other hand, there is a very graceful formulation [20–23] of the Dirac–Hestenes theory in terms of modern interpretation of spinors as minimal left ideals of Clifford algebras [24,25], a brief review of which we give in Section 2. In Section 3 after a short historical introduction, generalized Weierstrass formulae in $\mathbf{C}^4$ are rewritten in a spinor representation type form (matrix representation of a biquaternion algebra $\mathbf{C}_2 \cong \mathbf{M}_2(\mathbf{C})$) and are identified with the Dirac–Hestenes spinors, the matrix representation of which is also isomorphic to $\mathbf{M}_2(\mathbf{C})$. It allows to use a well-known relation between Dirac–Hestenes and Dirac spinors [23,26] (see also [27]) to establish a relation between Weierstrass–Konopelchenko coordinates for surfaces immersed into $\mathbf{C}^4$ and Dirac spinors. Integrable deformations of surfaces defined by a modified Veselov–Novikov equation and their relation with integrable deformations of Dirac field on surface are considered at the end of the Section 3.

2. Spinors as minimal left ideals of Clifford algebras

Let us consider a Clifford algebra $C\ell_{p,q}(V, Q)$ over a field $\mathbf{K}$ of characteristic 0 ($\mathbf{K} = \mathbf{R}, \mathbf{K} = \Omega = \mathbf{R} \oplus \mathbf{R}, \mathbf{K} = \mathbf{C}$), where V is a vector space endowed with a nondegenerate quadratic form

$$Q = x_1^2 + \cdots + x_p^2 - \cdots - x_{p+q}^2.$$

The algebra $C\ell_{p,q}$ is naturally $\mathbf{Z}_2$-graded. Let $C\ell^+_{p,q}$ (resp. $C\ell^-_{p,q}$) be a set consisting of all even (resp. odd) elements of algebra $C\ell_{p,q}$. The set $C\ell^+_{p,q}$ is a subalgebra of $C\ell_{p,q}$. It is obvious that $C\ell_{p,q} = C\ell^+_{p,q} \oplus C\ell^-_{p,q}$.

When n is odd, a volume element $\omega = \mathbf{e}_{12\ldots p+q}$ commutes with all elements of algebra $C\ell_{p,q}$ and therefore belongs to a center of $C\ell_{p,q}$. Thus, in the case of n is odd we have for a center

$$\mathbf{Z}_{p,q} = \begin{cases} \mathbf{R} \oplus i\mathbf{R} & \text{if } \omega^2 = -1, \\ \mathbf{R} \oplus e\mathbf{R} & \text{if } \omega^2 = +1, \end{cases} \tag{1}$$

where e is a double unit. In the case of n is even the center of $C\ell_{p,q}$ consists the unit of algebra.

Let $\mathbf{R}_{p,q} = C\ell_{p,q}(\mathbf{R}^{p,q}, Q)$ be a real Clifford algebra ($V = \mathbf{R}^{p,q}$ is a real space). Analogously, in the case of a complex space we have $\mathbf{C}_{p,q} = C\ell_{p,q}(\mathbf{C}^{p,q}, Q)$. Moreover, it is obvious that $\mathbf{C}_{p,q} \cong \mathbf{C}_n$, where $n = p + q$. Further, let us consider the following most important in physics Clifford algebras and their isomorhisms to matrix algebras:

quaternions	$\mathbf{R}_{0,2} = \mathbf{H}$,
biquaternions	$\mathbf{C}_2 = \mathbf{R}_{3,0} \cong \mathbf{M}_2(\mathbf{C})$,
space–time algebra	$\mathbf{R}_{1,3} \cong \mathbf{M}_2(\mathbf{H})$,
Dirac algebra	$\mathbf{C}_4 = \mathbf{R}_{4,1} \cong \mathbf{M}_4(\mathbf{C}) \cong \mathbf{M}_2(\mathbf{C}_2)$.

The identity $\mathbf{C}_2 = \mathbf{R}_{3,0}$ for a biquaternion algebra known in physics as a Pauli algebra is immediately obtained from the definition of the center of the algebra $C\ell_{p,q}$ (1). Namely, for $\mathbf{R}_{3,0}$ we have a volume element $\omega = \mathbf{e}_{123} \in \mathbf{Z}_{3,0} = \mathbf{R} \oplus \mathrm{i}\mathbf{R}$, since $\omega^2 = -1$. The identity $\mathbf{C}_4 = \mathbf{R}_{4,1}$ is analogously proved. The isomorphism $\mathbf{R}_{4,1} \cong \mathbf{M}_2(\mathbf{C}_2)$ is a consequence of an algebraic modulo 2 periodicity of complex Clifford algebras: $\mathbf{C}_4 \cong \mathbf{C}_2 \otimes \mathbf{C}_2 \cong \mathbf{C}_2 \otimes \mathbf{M}_2(\mathbf{C}) \cong \mathbf{M}_2(\mathbf{C}_2)$ [28–30].

The left (resp. right) ideal of algebra $C\ell_{p,q}$ is defined by the expression $C\ell_{p,q}e$ (resp. $eC\ell_{p,q}$), where e is an idempotent satisfying the condition $e^2 = e$. Analogously, a minimal left (resp. right) ideal is a set of type $I_{p,q} = C\ell_{p,q}e_{pq}$ (resp. $e_{pq}C\ell_{p,q}$), where e_{pq} is a primitive idempotent, i.e., $e^2_{pq} = e_{pq}$ and e_{pq} cannot be represented as a sum of two orthogonal idempotents, i.e., $e_{pq} \neq f_{pq} + g_{pq}$, where $f_{pq}g_{pq} = g_{pq}f_{pq} = 0$, $f^2_{pq} = f_{pq}$, $g^2_{pq} = g_{pq}$. In the general case a primitive idempotent has a form [20]

$$e_{pq} = \tfrac{1}{2}(1 + \mathbf{e}_{\alpha_1})\tfrac{1}{2}(1 + \mathbf{e}_{\alpha_2}) \cdots \tfrac{1}{2}(1 + \mathbf{e}_{\alpha_k}), \tag{2}$$

where $\mathbf{e}_{\alpha_1}, \ldots, \mathbf{e}_{\alpha_k}$ are commuting elements of the canonical basis of $C\ell_{p,q}$ such that $(\mathbf{e}_{\alpha_i})^2 = 1$, $(i = 1, 2, \ldots, k)$. The values of k are defined by a formula

$$k = q - r_{q-p}, \tag{3}$$

where r_i are the Radon–Hurwitz numbers, values of which form a cycle of the period 8:

$$r_{i+8} = r_i + 4. \tag{4}$$

The values of all r_i are

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

For example, let consider a minimal left ideal of the space–time algebra $\mathbf{R}_{1,3}$. The Radon–Hurwitz number for algebra $\mathbf{R}_{1,3}$ is equal to $r_{q-p} = r_2 = 2$, and therefore from (3) we have $k = 1$. The primitive idempotent of $\mathbf{R}_{1,3}$ has a form

$$e_{13} = \tfrac{1}{2}(1 + \mathbf{e}_0),$$

or $e_{13} = \frac{1}{2}(1 + \Gamma_0)$, where Γ_0 is a matrix representation of the unit $\mathbf{e}_0 \in \mathbf{R}_{1,3}$. Thus, a minimal left ideal of $\mathbf{R}_{1,3}$ is defined by the expression

$$I_{1,3} = \mathbf{R}_{1,3}\tfrac{1}{2}(1 + \Gamma_0). \tag{5}$$

Analogously, for the Dirac algebra $\mathbf{R}_{4,1}$ on using the recurrence formula (4) we obtain $k = 1 - r_{-3} = 1 - (r_5 - 4) = 2$, and a primitive idempotent of $\mathbf{R}_{4,1}$ may be defined as follows:

$$e_{41} = \tfrac{1}{2}(1 + \Gamma_0)\tfrac{1}{2}(1 + \mathrm{i}\Gamma_{12}), \tag{6}$$

where $\Gamma_{12} = \Gamma_1\Gamma_2$ and $\Gamma_i (i = 0, 1, 2, 3)$ are matrix representations of the units of $\mathbf{R}_{4,1} = \mathbf{C}_4$:

$$\Gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Further, for a minimal left ideal of Dirac algebra $I_{4,1} = \mathbf{R}_{4,1}\frac{1}{2}(1+\Gamma_0)\frac{1}{2}(1+\mathrm{i}\Gamma_{12})$ using the isomorphisms $\mathbf{R}_{4,1} = \mathbf{C}_4 = \mathbf{C} \otimes \mathbf{R}_{1,3} \cong \mathbf{M}_2(\mathbf{C}_2), \mathbf{R}^+_{4,1} \cong \mathbf{R}_{1,3} \cong \mathbf{M}_2(\mathbf{H})$ and also an identity $\mathbf{R}_{1,3}e_{13} = \mathbf{R}^+_{1,3}e_{13}$ [22,23] we have the following expression [27]:

$$\begin{aligned} I_{4,1} &= \mathbf{R}_{4,1}e_{41} = (\mathbf{C} \otimes \mathbf{R}_{1,3})e_{41} \cong \mathbf{R}^+_{4,1}e_{41} \cong \mathbf{R}_{1,3}e_{41} \\ &= R_{1,3}e_{13}\tfrac{1}{2}(1+\mathrm{i}\Gamma_{12}) = \mathbf{R}^+_{1,3}e_{13}\tfrac{1}{2}(1+\mathrm{i}\Gamma_{12}). \end{aligned} \tag{7}$$

Let $\Phi \in \mathbf{R}_{4,1} \cong \mathbf{M}_4(\mathbf{C})$ be a Dirac spinor and $\phi \in \mathbf{R}^+_{1,3} \cong \mathbf{R}_{3,0} = \mathbf{C}_2$ be a Dirac–Hestenes spinor. Then from (7) the relation immediately follows between spinors Φ and ϕ:

$$\Phi = \phi\tfrac{1}{2}(1+\Gamma_0)\tfrac{1}{2}(1+\mathrm{i}\Gamma_{12}). \tag{8}$$

Since $\phi \in \mathbf{R}^+_{1,3} \cong \mathbf{R}_{3,0}$, the Dirac–Hestenes spinor can be represented by a biquaternion number

$$\phi = a^0 + a^{01}\Gamma_{01} + a^{02}\Gamma_{02} + a^{03}\Gamma_{03} + a^{12}\Gamma_{12} + a^{13}\Gamma_{13} + a^{23}\Gamma_{23} + a^{0123}\Gamma_{0123}. \tag{9}$$

Or in the matrix representation

$$\phi = \begin{pmatrix} \phi_1 & -\phi_2^* & \phi_3 & \phi_4^* \\ \phi_2 & \phi_1^* & \phi_4 & -\phi_3^* \\ \phi_3 & \phi_4^* & \phi_1 & -\phi_2^* \\ \phi_4 & -\phi_3^* & \phi_2 & \phi_1^* \end{pmatrix}, \quad \phi_i \in \mathbf{C}, \tag{10}$$

where

$$\phi_1 = a^0 - \mathrm{i}a^{12}, \quad \phi_2 = a^{31} - \mathrm{i}a^{23}, \quad \phi_3 = a^{03} - \mathrm{i}a^{0123}, \quad \phi_4 = a^{01} + \mathrm{i}a^{02}.$$

Finally, from (8) it follows that for the Dirac spinor Φ and also a space–time spinor $Z = \phi\frac{1}{2}(1+\Gamma_0)$ we have expressions

$$\Phi = \begin{pmatrix} \phi_1 & 0 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 \\ \phi_3 & 0 & 0 & 0 \\ \phi_4 & 0 & 0 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} \phi_1 & -\phi_2^* & 0 & 0 \\ \phi_2 & \phi_1^* & 0 & 0 \\ \phi_3 & \phi_4^* & 0 & 0 \\ \phi_4 & -\phi_3^* & 0 & 0 \end{pmatrix},$$

which are minimal left ideals of algebras $\mathbf{R}_{4,1}$ and $\mathbf{R}_{1,3}$, respectively.

The Dirac spinor Φ may be considered as a vector in the 4-dimensional complex space $\mathbf{C}^4$ associated with the algebra $\mathbf{C}_4$. However, from a physical point of view it is more natural to consider the spinor Φ in space–time $\mathbf{R}^{1,3}$. In connection with this, let us introduce (following [21–23,26]) a more rigorous definition of spinor as a minimal left ideal of algebra $C\ell_{p,q}$.

Let $\mathfrak{B}_\Sigma = \{\Sigma_0, \dot{\Sigma}, \ddot{\Sigma}, \ldots\}$ be a set of all ordered orthonormal bases for $\mathbf{R}^{p,q}$. Any two bases $\Sigma_0, \dot{\Sigma} \in \mathfrak{B}_\Sigma$ are related by the element of the group $\mathrm{Spin}_+(p,q)$:

$$\dot{\Sigma} = u\Sigma_0 u^{-1}, \quad u \in \mathrm{Spin}_+(p,q).$$

Analogously, for the primitive idempotents defined in the basis $\Sigma \in \mathfrak{B}_\Sigma$ and denoted as $e_{\Sigma_0}, e_{\dot{\Sigma}}, \ldots$, we have $e_{\dot{\Sigma}} = ue_{\Sigma_0}u^{-1}$, $u \in \mathrm{Spin}_+(p,q)$. Then the ideals $I_{\Sigma_0}, I_{\dot{\Sigma}}, I_{\ddot{\Sigma}}, \ldots$ are geometrically equivalent if and only if

$$I_{\dot{\Sigma}} = uI_{\Sigma_0}u^{-1}, \quad u \in \mathrm{Spin}_+(p,q),$$

or, since $uI_{\Sigma_0} = I_{\Sigma_0}$:

$$I_{\dot{\Sigma}} = I_{\Sigma_0}u^{-1}.$$

Therefore, an algebraic spinor for $\mathbf{R}^{p,q}$ is an equivalence class of the quotient set $\{I_\Sigma\}/\mathbf{R}$, where $\{I_\Sigma\}$ is a set of all geometrically equivalent ideals, and $\Phi_{\Sigma_0} \in I_{\Sigma_0}$ and $\Phi_{\dot{\Sigma}} \in I_{\dot{\Sigma}}$ are equivalent, $\Phi_{\dot{\Sigma}} \cong \Phi_{\Sigma_0} (\mathrm{mod}\, \mathbf{R})$ if and only if $\Phi_{\dot{\Sigma}} = \Phi_{\Sigma_0}u^{-1}$, $u \in \mathrm{Spin}_+(p,q)$.

3. Weierstrass representation for surfaces in space $\mathbf{C}^4$

Historically, the Weierstrass representation [1] appeared in the result of the following variational problem: among the surfaces restricted by some curve for finding such a surface, the area of which is minimal, i.e., it is necessary to find a minimum of the functional

$$S = \int\int \sqrt{1 + p^2 + q^2}\, \mathrm{d}x\, \mathrm{d}y,$$

where $p = \mathrm{d}z/\mathrm{d}x$, $q = \mathrm{d}z/\mathrm{d}y$, $z = f(x,y)$ is an equation of the surface. The Euler equation for this problem has a form

$$\frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1+p^2+q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{q}{\sqrt{1+p^2+q^2}}\right) = 0.$$

This equation expresses a main geometrical property of such a surface: in each point the mean curvature is equal to zero. The surface which possesses such a property is called *a minimal surface*. If we compare a region $\mathfrak{M}$ of the surface with a region $\mathfrak{E}$ of the flat surface so that the point on $\mathfrak{M}$ with the coordinates (X^1, X^2, X^3) corresponds to a point $w = u + \mathrm{i}v$ of region $\mathfrak{E}$, then for the minimal surface we have the equations

$$\frac{\partial^2 X^1}{\partial u^2} + \frac{\partial^2 X^1}{\partial v^2} = 0, \qquad \frac{\partial^2 X^2}{\partial u^2} + \frac{\partial^2 X^2}{\partial v^2} = 0, \qquad \frac{\partial^2 X^3}{\partial u^2} + \frac{\partial^2 X^3}{\partial v^2} = 0,$$

solutions of which are of the form

$$X^1 = \operatorname{Re} f(w), \qquad X^2 = \operatorname{Re} g(w), \qquad X^3 = \operatorname{Re} h(w),$$

at

$$(f'(w))^2 + (g'(w))^2 + (h'(w))^2 = 0.$$

The functions satisfying this equation are

$$f'(w) = \mathrm{i}(G^2 + H^2), \qquad g'(w) = G^2 - H^2, \qquad h'(w) = 2GH,$$

where

$$\begin{aligned} X^1 &= C^1 + \operatorname{Re} \int_{w_0}^{w} \mathrm{i}(G^2 + H^2)\,\mathrm{d}w, \\ X^2 &= C^2 + \operatorname{Re} \int_{w_0}^{w} (G^2 - H^2)\,\mathrm{d}w, \\ X^3 &= C^3 + 2\operatorname{Re} \int_{w_0}^{w} GH\,\mathrm{d}w. \end{aligned} \tag{11}$$

Here $G(w)$ and $H(w)$ are holomorphic functions defined in a circle or in all complex plane. After substitution of variables,

$$s = \xi + \mathrm{i}\eta = \frac{H(w)}{G(w)}, \qquad G^2 \frac{\mathrm{d}w}{\mathrm{d}s} = F(s),$$

the equations (11) take the form

$$\begin{aligned} \mathrm{d}X^1 &= \operatorname{Re}[\mathrm{i}(1 + s^2)F(s)\,\mathrm{d}s], \\ \mathrm{d}X^2 &= \operatorname{Re}[(1 - s^2)F(s)\,\mathrm{d}s], \\ \mathrm{d}X^3 &= \operatorname{Re}[2sF(s)\,\mathrm{d}s]. \end{aligned}$$

Thus, for an every analytic function $F(s)$ we have a minimal surface.

Further, let us consider generalized Weierstrass representation for surfaces immersed into 4-dimensional complex space $\mathbf{C}^4$, which, as known, is associated with the Dirac algebra $\mathbf{C}_4$. In this case generalized Weierstrass formulae have a form

$$
\begin{aligned}
X^1 &= \frac{\mathrm{i}}{2}\int_\Gamma (\psi_1\psi_2\,\mathrm{d}\bar{z} - \varphi_1\varphi_2\,\mathrm{d}z),\\
X^2 &= \frac{1}{2}\int_\Gamma (\psi_1\psi_2\,\mathrm{d}\bar{z} + \varphi_1\varphi_2\,\mathrm{d}z),\\
X^3 &= \frac{1}{2}\int_\Gamma (\psi_1\varphi_2\,\mathrm{d}\bar{z} - \varphi_1\psi_2\,\mathrm{d}z),\\
X^4 &= \frac{\mathrm{i}}{2}\int_\Gamma (\psi_1\varphi_2\,\mathrm{d}\bar{z} + \varphi_1\psi_2\,\mathrm{d}z),
\end{aligned} \tag{12}
$$

$$
\psi_{\alpha z} = p\varphi_\alpha, \quad \varphi_{\alpha\bar{z}} = -p\psi_\alpha, \qquad \alpha = 1, 2, \tag{13}
$$

where ψ, φ and p are complex-valued functions on variables $z, \bar{z} \in \mathbf{C}$, Γ is a contour in complex plane $\mathbf{C}$. We will interpret the functions $X^i(z,\bar{z})$ as the coordinates in $\mathbf{C}^4$. It is easy to verify that components of an induced metric have a form

$$
g_{zz} = \overline{g_{\bar{z}\bar{z}}} = \sum_{i=1}^{4}(X^i_z)^2 = 0,
$$

$$
g_{z\bar{z}} = \sum_{i=1}^{4}(X^i_z X^i_{\bar{z}}) = \psi_1\psi_2\varphi_1\varphi_2.
$$

Therefore, the formulae (12) and (13) define a conformal immersion of the surface into $\mathbf{C}^4$ with an induced metric

$$
\mathrm{d}s^2 = \psi_1\psi_2\varphi_1\varphi_2\,\mathrm{d}z\,\mathrm{d}\bar{z}.
$$

The formulae (12) may be rewritten in the following form

$$
\begin{aligned}
\mathrm{d}(X^1 + \mathrm{i}X^2) &= \mathrm{i}\psi_1\psi_2\,\mathrm{d}\bar{z},\\
\mathrm{d}(X^1 - \mathrm{i}X^2) &= -\mathrm{i}\varphi_1\varphi_2\,\mathrm{d}z,\\
\mathrm{d}(X^4 + \mathrm{i}X^3) &= \mathrm{i}\psi_1\varphi_2\,\mathrm{d}\bar{z},\\
\mathrm{d}(X^4 - \mathrm{i}X^3) &= \mathrm{i}\varphi_1\psi_2\,\mathrm{d}z,
\end{aligned}
$$

or

$$
\mathrm{d}(X^4\sigma_0 + X^1\sigma_1 + X^2\sigma_2 + X^3\sigma_3) = \mathrm{i}\begin{pmatrix} \varphi_1\psi_2\,\mathrm{d}z & \psi_1\psi_2\,\mathrm{d}\bar{z} \\ \varphi_1\varphi_2\,\mathrm{d}z & \psi_1\varphi_2\,\mathrm{d}\bar{z} \end{pmatrix}, \tag{14}
$$

where

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}
$$

are matrix representations of the units of quaternion algebra $\mathbf{R}_{0,2} = \mathbf{H} : \mathrm{e}_i \longrightarrow \sigma_i (i = 0, 1, 2)$, $\mathrm{e}_{21} \longrightarrow \sigma_3$. It is easy to see that the left part of the expression (14) is a biquaternion $\mathbf{C}_2 = \mathbf{C} \otimes \mathbf{R}_{0,2}$. Recalling that $\mathbf{C}_2 = \mathbf{R}_{3,0}$ and a volume element $\omega = \mathbf{e}_{123} \in \mathbf{R}_{3,0}$ belongs to a center $\mathbf{Z}_{3,0} = \mathbf{R} \oplus \mathrm{i}\mathbf{R}$, we can write a biquaternion $X^4\mathbf{e}_0 + X^1\mathbf{e}_1 + X^2\mathbf{e}_2 + X^3\mathbf{e}_3$, where $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{e}_3 = \mathbf{e}_{21} = \mathbf{e}_2\mathbf{e}_1$, in the form

$$\begin{aligned}
&\mathrm{Re}X^4\mathbf{e}_0 + \mathrm{Re}X^1\mathbf{e}_1 + \mathrm{Re}X^2\mathbf{e}_2 + \mathrm{Re}X^3\mathbf{e}_3 \\
&\quad + \mathrm{Im}\,X^3\mathbf{e}_{12} + \mathrm{Im}\,X^2\mathbf{e}_{31} + \mathrm{Im}\,X^1\mathbf{e}_{23} + \mathrm{Im}\,X^4\mathbf{e}_{123} \\
&\quad\quad = (\mathrm{Re}X^4 + \omega\,\mathrm{Im}\,X^4)\mathbf{e}_0 + (\mathrm{Re}X^1 + \omega\,\mathrm{Im}\,X^1)\mathbf{e}_1 \\
&\quad\quad\quad + (\mathrm{Re}X^2 + \omega\,\mathrm{Im}\,X^2)\mathbf{e}_2 + (\mathrm{Re}X^3 + \omega\,\mathrm{Im}\,X^3)\mathbf{e}_3 \\
&\quad\quad = X^4\mathbf{e}_0 + X^1\mathbf{e}_1 + X^2\mathbf{e}_2 + X^3\mathbf{e}_3.
\end{aligned} \tag{15}$$

Further, by means of isomorphisms $\mathbf{R}_{3,0} \cong \mathbf{R}^+_{1,3}$ and $\mathbf{R}^{++}_{4,1} \cong \mathbf{R}^+_{1,3} \cong \mathbf{R}_{3,0}$ the biquaternion (15) may be rewritten as (like (9)):

$$\begin{aligned}
\phi = {}&\mathrm{Re}X^4 I + \mathrm{Re}X^1\Gamma_{01} + \mathrm{Re}X^2\Gamma_{02} + \mathrm{Re}X^3\Gamma_{03} \\
&+ \mathrm{Im}\,X^3\Gamma_{12} + \mathrm{Im}\,X^2\Gamma_{31} + \mathrm{Im}\,X^1\Gamma_{23} + \mathrm{Im}\,X^4\Gamma_{0123}.
\end{aligned} \tag{16}$$

Or in the form (10) if suppose

$$\begin{aligned}
\phi_1 &= \mathrm{Re}X^4 - \mathrm{i}\,\mathrm{Im}\,X^3, \\
\phi_2 &= \mathrm{Im}\,X^2 - \mathrm{i}\,\mathrm{Im}\,X^1, \\
\phi_3 &= \mathrm{Re}X^3 - \mathrm{i}\,\mathrm{Im}\,X^4, \\
\phi_4 &= \mathrm{Re}X^1 + \mathrm{i}\mathrm{Re}X^2.
\end{aligned} \tag{17}$$

The formulae (17) define a relation between Weierstrass–Konopelchenko coordinates and Dirac–Hestenes spinors. This relation is a direct consequence of an isomorphism $\mathbf{C}_2 = \mathbf{C} \otimes \mathbf{R}_{0,2} = \mathbf{R}_{3,0} \cong \mathbf{R}^+_{1,3}$. Further, using the idempotent $\frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + \mathrm{i}\Gamma_{12})$ it is easy to establish (by means of (8)) a relation with the Dirac spinor treated as a minimal left ideal of algebra $\mathbf{R}_{4,1} = \mathbf{C}_4 \cong \mathbf{M}_4(\mathbf{C})$:

$$\Phi = \begin{pmatrix} \phi_1 & 0 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 \\ \phi_3 & 0 & 0 & 0 \\ \phi_4 & 0 & 0 & 0 \end{pmatrix}.$$

It is obvious that we cannot directly identify the spinor defined by the formulae (17) with a generic "physical" spinor of electron theory, because in accordance with (17) and (12)–(13) the spinor ϕ depends only on two variables $z, \bar{z}$, or x^1, x^2 if suppose $z = x^1 + \mathrm{i}x^2$, whilst a physical spinor with four components depends on four variables x^1, x^2, x^3, x^4. By this reason we will call the spinor defined by the identities (17) as *a surface spinor*, and respectively the field $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^{\mathrm{T}}$ will be called *a Dirac spinor field on surface*. The relationship between a surface spinor $\phi(z, \bar{z})$ and a physical (space) spinor $\phi(x^1, x^2, x^3, x^4)$, and also a relation with the spinor representations of surfaces in spaces $\mathbf{R}^{p,q}$ will be considered in a separate paper.

Further, according to Section 2, an algebraic Dirac spinor for $\mathbf{R}^{1,3}$ is an element of $\{I_\Sigma\}/\mathbf{R}$. Then if $\Phi_{\Sigma_0} \in I_{\Sigma_0}$, $\Phi_{\dot{\Sigma}} \in I_{\dot{\Sigma}}$, then $\Phi_{\dot{\Sigma}} \simeq \Phi_{\Sigma_0} (\mathrm{mod}\,\mathbf{R})$ if and only if

$$\Phi_{\dot{\Sigma}} = \Phi_{\Sigma_0} u^{-1}, \quad u \in \mathrm{Spin}_+(1, 3). \tag{18}$$

Here in accordance with (8)

$$\Phi_{\Sigma_0} = \phi_{\Sigma_0}\tfrac{1}{2}(1+\Gamma_0)\tfrac{1}{2}(1+\mathrm{i}\Gamma_{12}).$$

The formula (18) defines a transformation law of the Dirac spinor. It is obvious that a transformation group of the biquaternion (14) is also isomorphic to $\mathrm{Spin}_+(1,3)$, since

$$\mathrm{Spin}_+(1,3) \cong \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathbf{C}_2 \colon \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1 \right\} = \mathrm{SL}(2;\mathbf{C}),$$

where $\mathrm{SL}(2;\mathbf{C})$ is a double covering of the own Lorentz group $\pounds_+^{\uparrow}$. Therefore, *the transformations of Weierstrass–Konopelchenko coordinates for surfaces immersed into* $\mathbf{C}^4$ *are induced (via the relations* (17)) *transformation of a Dirac field* $\Phi = (\phi_1,\phi_2,\phi_3,\phi_4)^{\mathrm{T}}$ *in* $\mathbf{R}^{1,3}$, *where* $\Phi \in \mathbf{M}_4(\mathbf{C})e_{41}$ *is the minimal left ideal of* $\mathbf{R}_{4,1} \cong \mathbf{M}_4(\mathbf{C})$ *defined in some orthonormal basis* $\Sigma \in \mathfrak{B}_\Sigma$.

On the other hand, if suppose (following [2,3,15]) that the functions p, ψ_α and φ_α in (13) depend on the deformation parameter t, then the deformations of ψ_α and φ_α are defined by the following system:

$$\begin{aligned} \psi_{\alpha t} &= A\psi_\alpha + B\varphi_\alpha, \\ \varphi_{\alpha t} &= C\psi_\alpha + D\varphi_\alpha, \end{aligned} \qquad \alpha = 1,2, \tag{19}$$

where A, B, C, D are differential operators. The equations (19) define integrable deformations of surfaces immersed in $\mathbf{C}^4$. Let p be a real-valued function; the compatibility condition of (19) with (13) is equivalent to the nonlinear partial differential equation for p. In the simplest nontrivial case (A, B, C, D are first order operators) it is a modified Veselov–Novikov equation [31]:

$$\begin{aligned} &p_t + p_{zzz} + p_{\bar z\bar z\bar z} + 3p_z\omega + 3p_{\bar z}\bar\omega + \tfrac{3}{2}p\bar\omega_{\bar z} + \tfrac{3}{2}p\omega_z = 0, \\ &\omega_{\bar z} = (p^2)_z. \end{aligned}$$

Varying operators A, B, C, D one gets an infinite hierarchy of integrable equations for p (modified Veselov–Novikov hierarchy [32,33,31,2]). It is obvious that the deformation of $\psi_\alpha, \varphi_\alpha$ via (19) induced the deformations of the coordinates $X^i(z,\bar z,t)$ in $\mathbf{C}^4$. Moreover, according to (14) and (16) treated as a matrix representation of the Dirac–Hestenes spinor field ϕ, we may say that the mVN-deformation generates a deformation of the Dirac field $\Phi = (\phi_1,\phi_2,\phi_3,\phi_4)^{\mathrm{T}}$.

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References

[1] K. Weierstrass, Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich Null ist, Monatsber. Akad. Wiss. Berlin (1866) 612–625.
[2] B.G. Konopelchenko, Induced surfaces and their integrable dynamics, Stud. Appl. Math. 96 (1996) 9–51; preprint, Institute of Nuclear Physics, N 93-144, Novosibirsk, 1993.
[3] B.G. Konopelchenko, Multidimensional integrable systems and dynamics of surfaces in space, preprint, Institute of Mathematics, Taipei, August 1993.
[4] R. Carroll, B.G. Konopelchenko, Generalized Weierstrass–Enneper inducing, conformal immersion and gravity, Int. J. Modern Phys. A 11 (1996) 1183–1216.
[5] B.G. Konopelchenko, I.A. Taimanov, Generalized Weierstrass formulae, soliton equations and Willmore surfaces, preprint N.187, Univ. Bochum, 1995.
[6] B.G. Konopelchenko, I.A. Taimanov, Constant mean curvatire surfaces via an integrable dynamical system, J. Phys. A 29 (1996) 1261–1265.
[7] I.A. Taimanov, Modified Novikov-Veselov equation and differential geometry of surfaces, Trans. Amer. Math. Soc., Ser. 2, 179 (1997) 133–159.
[8] I.A. Taimanov, Surfaces of revolution in terms of solitons, Ann. Glob. Anal. Geom. 15 (5) (1997) 419–435.
[9] I.A. Taimanov, The Weierstrass representation of closed surfaces in R^3, preprint dg-ga/9710020 (1997).
[10] I.A. Taimanov, Global Weierstrass representation and its spectrum, Uspechi Mat. Nauk 52 (6) (1997) 187–188.
[11] P.G. Grinevich, M.V. Schmidt, Conformal invariant functionals of immersion of tori into R^3, J. Geomet. and Phys. 26 (1–2) (1998) 51–78.
[12] B.G. Konopelchenko, G. Landolfi, On classical string configurations, Mod. Phys. Lett. A 12 (1997) 3161–3168.
[13] R. Kusner, N. Schmitt, The spinor representation of surfaces in space, preprint dg-ga/9610005 (1996).
[14] T. Friedrich, On the spinor representation of surfaces in Euclidean 3-spaces, J. Geom. Phys. 28 (1998) 143–157.
[15] B.G. Konopelchenko, G. Landolfi, Generalized Weierstrass representation for surfaces in multidimensional Riemann spaces, preprint math. DG/9804144, 1998; J. Geomet. and Phys. 29 (1999) 319–333.
[16] D. Hestenes, Real spinor fields, J. Math. Phys. 8 (1967) 798–808.
[17] D. Hestenes, Observables, operators, and complex numbers in the Dirac theory, J. Math. Phys. 16 (1976) 556–571.
[18] D. Hestenes, Space-Time Algebra, Gordon and Breach, New York, 1987.
[19] D. Hestenes, G. Sobczyk, Clifford algebra to Geometrical Calculus, D. Heidel Publ., Dordrecht, 1984.
[20] P. Lounesto, Scalar product of spinors and an extension of Brauer-Wall groups, Found. Phys. 11 (1981) 721–740.
[21] P. Lounesto, Clifford algebras and Hestenes spinors, Found. Phys. 23 (1993) 1203–1237.
[22] V.L. Figueiredo, W.A. Rodrigues Jr., E.C. Oliveira, Covariant, algebraic, and operator spinors, Int. J. Theor. Phys. 29 (1990) 371–395.
[23] V.L. Figueiredo, W.A. Rodrigues Jr., E.C. Oliveira, Clifford algebras and the hidden geometrical nature of spinors, Algebras, Groups and Geometries 7 (1990) 153–198.
[24] A. Crumeyrolle, The primitive idempotents of the Clifford algebras and the amorphic spinor fibre bundles, Reports on Math. Phys. 25 (1987) 305–328.
[25] A. Crumeyrolle, Orthogonal and Symplectic Clifford Algebras, Kluwer Academic Publishers, Dordrecht, 1991.
[26] W.A. Rodrigues Jr., Q.A.G. De Souza, J. Vaz Jr., P. Lounesto, Dirac–Hestenes spinor fields in Riemann-Cartan spacetime, Int. J. Theor. Phys. 35 (1996) 1849–1900.
[27] J. Vaz Jr., W.A. Rodrigues Jr., Maxwell and Dirac theories as an already unified theory, preprint hep-th/9511181, to appear in: J. Keller, Oziewicz (Eds.), Proceedings of the International Conference on the Theory of the Electron, UNAM, Mexico, 1995.
[28] M.F. Atiyah, R. Bott, A. Shapiro, Clifford modules, Topology 3 (Suppl. 1) (1964) 3–38.
[29] I. Porteous, Topological Geometry, van Nostrand, London, 1969.
[30] M. Karoubi, K-Theory, Spinger, Berlin, 1979.

[31] L.V. Bogdanov, Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation, Theor. Math. Phys. 70 (1987) 309–314.
[32] A.P. Veselov, S.P. Novikov, Finite-zone, two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations, Soviet Math. Dokl. 30 (1984) 588–591.
[33] A.P. Veselov, S.P. Novikov, Finite-zone, two-dimensional Schrödinger operators. Potential operators, Soviet Math. Dokl. 30 (1984) 705–708.